Modelling time-dependent processes resulting from thermo-viscoelastic behaviour of rocks

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ABSTRACT: In rock mechanics, various situations are known where one may need a viscoelastic – rather than elastic – modelling. Examples include laboratory cyclic loading experiments, the Anelastic Strain Recovery method for determining in situ stress, and a recent new such area is the gravitational effect of waves (seemingly small but actually considerably relevant mass rearrangements) around underground gravitational wave detectors. It is also known that temperature influences the mechanical processes through thermal expansion, modifying strains and displacements and inducing thermal stresses. In certain cases, heat conduction – which determines the distribution of temperature – must also be incorporated in the description. Modelling these aspects poses a serious challenge. We present a thermodynamically consistent framework with a viscoelastic material model that captures these complex, coupled phenomena in a unified way. Our corresponding self-developed numerical scheme is able to predict the aforementioned thermo-viscoelastic effects for various scenarios reliably, with low resource demand.

Keywords: viscoelasticity, thermal effects, wave propagation, extended symplectic numerics.

1 INTRODUCTION

Investigating measured time dependence of stress, strain and displacement reveals that rocks exhibit rate-dependent behavior – including, among others, the observation that dynamic elastic moduli are higher than their static counterparts (Davarpanah et al. 2020) –, which raises the need for viscoelastic modelling (Asszonyi et al. 2015). Apart from a limited range of problems where analytical approaches are feasible (see, e.g., Fülöp & Béda 2010, Fülöp & Szücs 2020, Fülöp & Szücs 2022), this in itself raises the need for numerical solutions. Experience also shows that thermal expansion may considerably influence strains and, in such cases, a rheologically extended thermoelastic description is required – for example, for the Anelastic Strain Recovery method (Matsuki & Takeuchi 1993, Matsuki 2008, Lin et al. 2010). The involved temperature field is influenced by heat conduction, which might also need to be coupled. These altogether demand a reliable numerical solver.

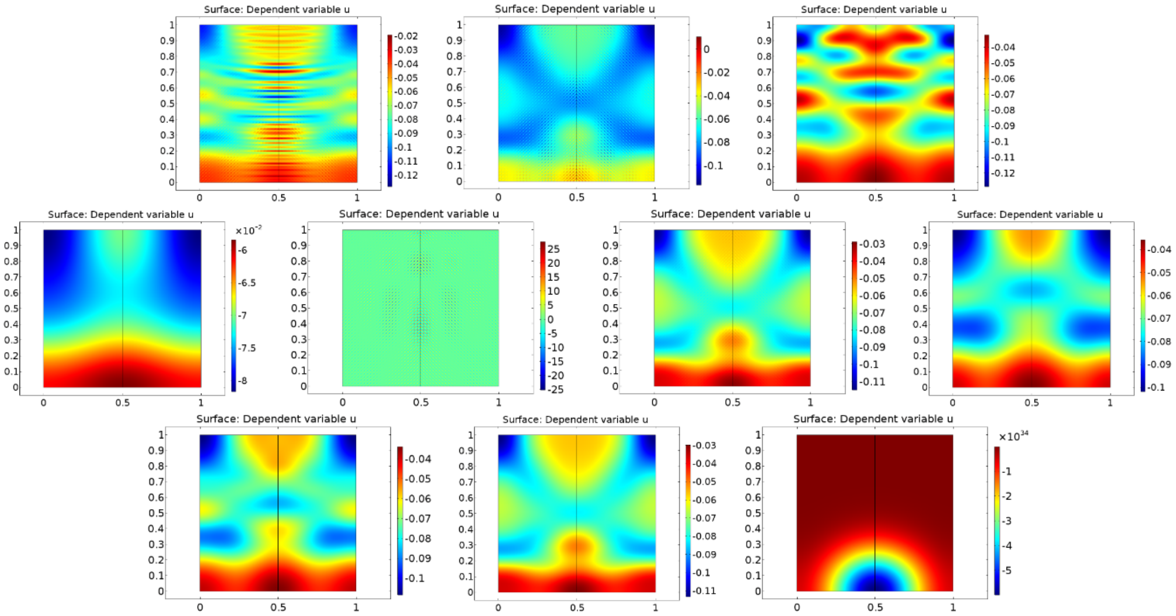

Figure 1. Purely elastic wave propagation in two dimensions, snapshot of the distribution of displacement, the same problem solved with ten different settings in the finite-element software COMSOL (Pozsár et al. 2020). Which outcome may be the closest to the correct solution pattern?

Our experience has shown that standard commercial software solutions are not suitable for such tasks (see Figure 1). This has motivated us to develop an own finite-difference approach (Fülöp et al. 2020, Pozsár et al. 2020, Fülöp 2021), which is a thermodynamically consistent extension of a symplectic scheme. For reversible systems, symplectic schemes (see, e.g., Hairer 2006, Denker 2021) possess outstanding properties, and our extension has successfully preserved much of them.

2 CONTINUUM MODEL AND DISCRETIZATION IN ONE DIMENSION

For a one-dimensional setup, our set of equations to solve is as follows.

$$\varrho \dot{v} = \frac{\partial \sigma}{\partial x}, \qquad \sigma = \sigma_{\mathrm{el}} + \hat{\sigma}, \qquad \sigma_{\mathrm{el}} = ED, \qquad \hat{\sigma} + \tau \frac{\partial \hat{\sigma}}{\partial t} = \hat{I} L, \tag{1}$$

$$L = \frac{\partial v}{\partial x}, \qquad \epsilon = D + \alpha (T - T_{\mathrm{ex}}), \qquad \frac{\partial \epsilon}{\partial t} = L, \tag{2}$$

$$\varrho \frac{\partial e_{\mathrm{int}}}{\partial t} = -\frac{\partial j_e}{\partial x} + \sigma L, \qquad \varrho \frac{\partial s}{\partial t} = -\frac{\partial j_s}{\partial x} + \pi_s, \qquad s = c_{\mathrm{ex}} \ln \frac{T}{T_{\mathrm{ex}}} + \frac{E\alpha}{\varrho} \epsilon, \tag{3}$$

$$j_e = -\lambda \frac{\partial T}{\partial x}, \qquad j_s = \frac{1}{T} j_e, \qquad \pi_s = \frac{\partial \frac{1}{T}}{\partial x} j_e + \frac{1}{\hat{I}} \frac{1}{T} \hat{\sigma} \left(\hat{I} L - \tau \frac{\partial \hat{\sigma}}{\partial t} \right), \tag{4}$$

$$e_{\mathrm{total}} = e_{\mathrm{kin}} + e_{\mathrm{int}}, \qquad e_{\mathrm{kin}} = \frac{v^2}{2}, \qquad e_{\mathrm{int}} = e_{\mathrm{therm}} + e_{\mathrm{el}} + e_{\mathrm{rheol}}, \tag{5}$$

$$e_{\text{therm}} = c_{\text{ex}}(T - T_{\text{ex}}), \qquad e_{\text{el}} = \frac{E}{2\varrho}\epsilon(\epsilon + \alpha T_{\text{ex}}), \qquad e_{\text{rheol}} = \frac{\tau}{2\varrho \hat{I}}\hat{\sigma}^2. \tag{6}$$

Here ϱ, E, c_{ex}, λ, α, τ, $\hat{I}$ denote the density, Young's modulus, specific heat capacity, heat conduction coefficient, linear thermal expansion coefficient, relaxation time, and index-of-damping material parameters, respectively. Additionally, v denotes the velocity field, L the velocity gradient, σ the Cauchy stress decomposed into a σ_{el} elastic and a $\hat{\sigma}$ irreversible part, D the elastic deformedness or elastic strain (Nowacki 1986 and Hetnarski & Eslami 2009; see also Lubarda 2004), ϵ the total strain, T the absolute temperature, T_{ex} the expansion-point reference temperature, j_e the heat current density, s the specific entropy, π_s the entropy production rate density, and j_s the entropy current density, respectively. Finally, e_{total} is the total specific energy, which is the sum of the e_{kin} specific kinetic, and e_{int} specific internal energies, the latter of which being equal to the sum of the e_{therm} specific thermal, e_{el} specific elastic, and e_{rheol} specific rheological energies.

The extended symplectic finite-difference scheme for these continuum equations can be summarized as shown in Figure 2. Solutions prove reliable, as demonstrated in Figures 3 and 4.

t \ x	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$\cdots$	$n-\frac{1}{2}$	n	$n+\frac{1}{2}$	$\cdots$
$-\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
0	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$+\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
$j-\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
j	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$j+\frac{1}{2}$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$	v, j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	v, j_e	$\cdots$
$j+1$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$	j_e	D, ϵ, σ, $\hat{\sigma}$, σ_{el}, e_{int}, T, L	j_e	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Figure 2. The finite-difference scheme. Grey quantities are auxiliary, not final solution values.

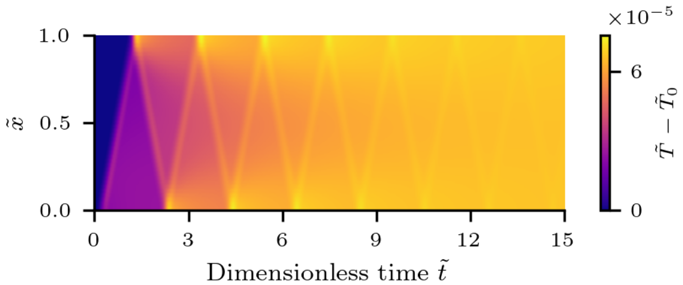

Figure 3. The distribution of temperature as the function of time along a one-dimensional sample after a single-pulse excitation imposed at one end. Temperature increase is due to viscoelastic dissipation, and homogenization of temperature is caused by heat conduction.

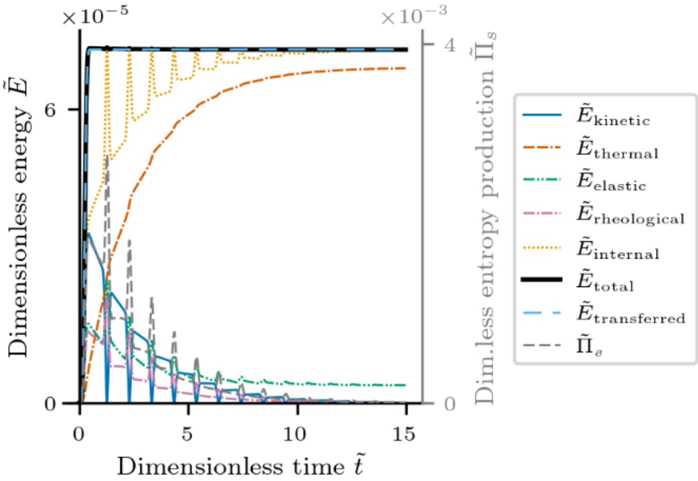

Figure 4. Plotting the various energies as the function of time proves to be a suitable means for monitoring the quality of the solution. The numerical scheme preserves total energy satisfactorily.

3 EXTENSION TO THREE-DIMENSIONAL CYLINDRICAL SETTINGS

The extension of the scheme to three-dimensional cylindrical sample geometries uses the discretization pattern as shown in Figure 5. As application, one of our motivations has been to simulate Resonant Frequency Method (see, e.g., Malhotra & Carino 2004) outcomes as visible in Figure 6. The numerical results prove to successfully produce the typical observed patterns, as illustrated in Figure 7.

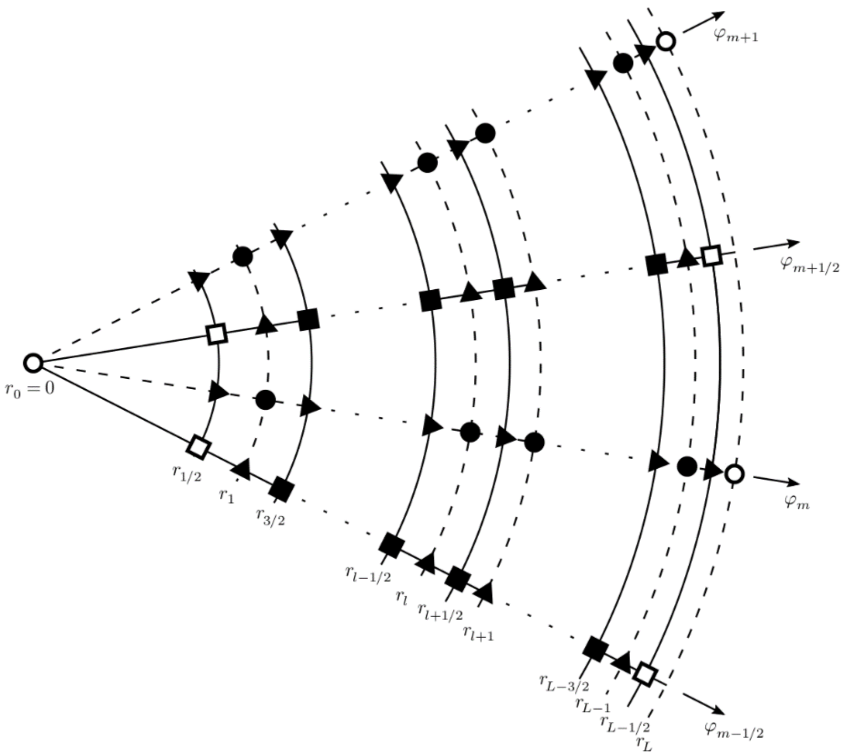

Figure 5. Velocity values (triangles), diagonal stress and strain values (circles), and offdiagonal stress and strain values (squares) in the cylindrical discretization. Void symbols represent boundary-condition values.

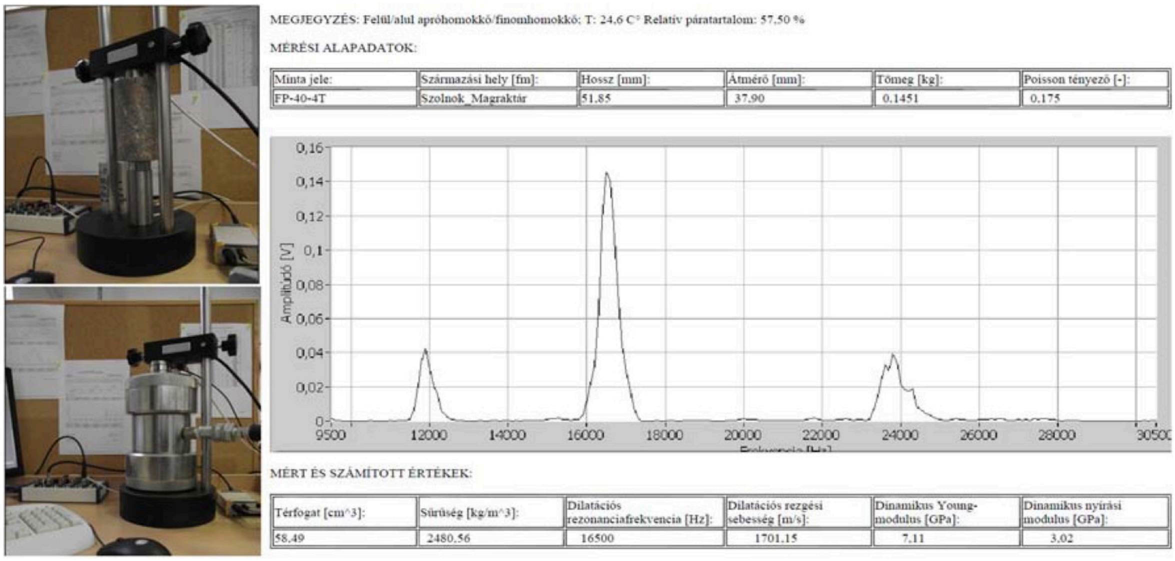

Figure 6. Device (left) and typical outcome (right) of the Resonant Frequency Method (Kovács et al. 2015).

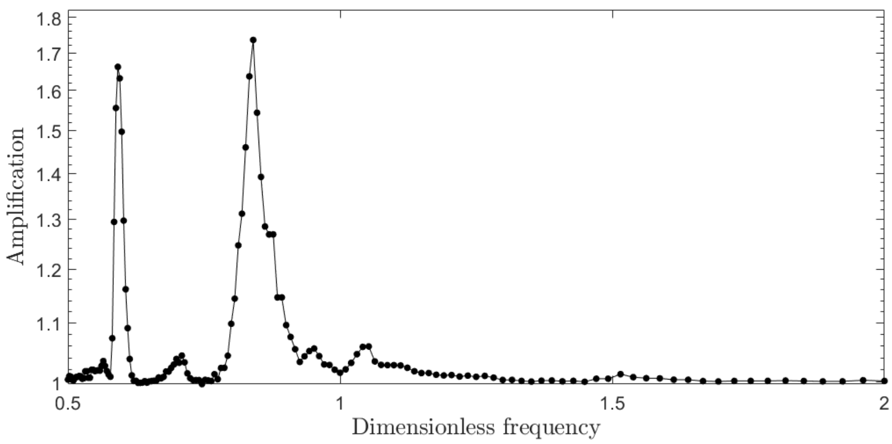

Figure 7. Simulation result for the Resonant Frequency Method.

4 CONCLUSION

The self-developed, explicit finite-difference extended symplectic numerical scheme for the heat-conduction coupled thermo-viscoelastic model yields reliable predictions, with low resource demand. Planned future application areas are simulations for laboratory cyclic loading experiments, higher-precision evaluation for the Anelastic Strain Recovery method, exploitation of the Resonant Frequency Method for determining viscoelastic material parameters, and the gravitational effect (the so-called Newtonian noise) of waves around underground gravitational wave detectors.

ACKNOWLEDGEMENTS

This work has been supported by the National Research, Development and Innovation Office–NKFIH grant FK 134277, the R&D project NO. 2018-1.1.2-KFI-2018-00207, the ÚNKP-22-3-I-BME-96 New National Excellence Program of the Ministry for Culture and Innovation from the source of the National Research, Development and Innovation Fund, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

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