Determining the transversely isotropic elastic constants from strain data by means of different mathematical approaches

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ABSTRACT: Mechanical properties of rocks with marked weak planes such as foliation, schistosity or bedding, are not isotropic. The deformability of these rocks is transversely anisotropic (TI), and, therefore, different axial strains are observed according to the load direction, being equal in directions within the weakness plane. Their elastic behavior is theoretically defined by five independent constants. In this study, a relatively large number of strain data sets collected from compressive tests on slate are used to estimate these constants. Two main approaches are followed. The first consists in solving the corresponding compliance matrix equation by means of the least square best fit multi-regression approach in groups of samples, which are then averaged. The second is based on minimizing the error in the stress invariant by means of a non-linear gradient algorithm. Results are analyzed aiming to learn how to manage strain variability in the process of TI elastic parameter estimate.

Keywords: Transversely isotropic elasticity, Elastic constants, Slate, GRG algorithm.

1 INTRODUCTION

Determining the elastic constants is important to understand and model the behavior of rocks and rock masses. A number of common metamorphic and sedimentary rocks such as shales, schists, slates or gneisses present marked foliation, so they are transversely isotropic materials, whose behavior is controlled by the orientation of these weakness planes. In these materials, the elastic behavior changes according to the direction in which a load is applied in relation to the transverse anisotropy plane. This behavior can be theoretically defined by five independent elastic parameters, namely two Young's moduli (E and E'), two Poisson's ratio (ν and ν') and a shear modulus (G).

Analytical methods based on Lekhnistskii's (1963) elasticity analyses can be used to obtain these parameters, as those proposed by Barla (1974) or Amadei (1982). To determine these elastic constants, several authors proposed theoretical approaches (Barla 1974, Amadei 1996, Chen et al 1998). The number of different oriented cores (typically two or three) should be adapted to the number of available equations to obtain the five elastic constants. If more strain data are available, the least square approach was recommended by Amadei (1996) to calculate the elastic constants that the best fitted test results.

These studies report there is a minimum number of samples needed to obtain the 5 elastic constants. Being θ the angle formed by the foliation and the horizontal plane in the samples, Amadei (1996) suggests that 3 different samples, one with $\theta = 0°$, the second $\theta = 90°$ and the third $0° < \theta < 90°$, can be used to obtain the five constants. Cho et al (2012) suggested to obtain the five elastic constants based on only two sample results, one with $\theta = 0°$ and the second with $0° < \theta < 90°$.

Although these are a rigorous theoretical approach, it is important to account for variability of actual strain results that can induce relevant errors when processing the data. Alejano et al (2021) proposed another approach based on minimizing the error of the strain invariant of all the available measurements with the help of a generalized reduced gradient (GRG) method.

2 METHODOLOGIES

In this study, the five elastic constants were calculated by using two different methods. Both methods are derived from the equations proposed by Amadei (1982), using the Generalized Hooke´s law, in which the 21 elastic constants are reduced to five due to the symmetry existing in the transversely isotropic rocks. The generalized Hooke's law in global coordinates can be written in matrix terms as (Barla, 1974):

$$\varepsilon' = S'\sigma' \tag{1}$$

Operating the Generalized Hooke's matrix for conveniently applied loads (uniaxial compression) and adequate strain measurements, and applying symmetry relations (eqs. 2, 3 and 4; Lekhnistskii, 1963) corresponding strains in the axis directions with the corresponding applied stresses are obtained:

$$\frac{\Delta\varepsilon_y}{\Delta\sigma_y} = \frac{sin^4\theta}{E} + \frac{cos^4\theta}{E'} + \frac{sin^2 2\theta}{4}\left(\frac{-2v'}{E'} + \frac{1}{G'}\right) \tag{2}$$

$$\frac{\Delta\varepsilon_x}{\Delta\sigma_y} = \frac{sin^2 2\theta}{4}\left(\frac{1}{E} + \frac{1}{E'} - \frac{1}{G'}\right) - \frac{v'}{E'}(cos^4\theta + sin^4\theta) \tag{3}$$

$$\frac{\Delta\varepsilon_z}{\Delta\sigma_y} = -sin^2\theta\frac{v}{E} - cos^2\theta\frac{v'}{E'} \tag{4}$$

where θ is the angle between the horizontal axis x and the transversely isotropic plane, as illustrated in Fig. 1. Typically samples with $\theta = 0º$ (Fig. 1.a), $\theta = 90º$ (Fig. 1 b.) and $0º < \theta < 90º$ (i.e., 15º, 30º, 45º, 60º, as in Fig. 1.c and 75º) are tested. These equations are valid assuming uniform stresses and strains in the tested specimens. Strains must be measured in adequate, well-defined directions (Fig. 1.d). These equations can be extended from uniaxial to triaxial tests, provided confinement stress is kept constant in the range of elastic strain measurements.

Based in Cho et al. (2012), it is possible to adapt matrix equation systems that can be applied to different groups of tests, as those presented in Fig. 2, for the case of groups of 3 and 4 particular types of the samples (one $\theta = 0º$ and the others with $\theta \neq 0º$). According to Amadei (1982, 1996), these equation systems can be solved by least square multilinear regression analysis to estimate the five compliances. Having a sufficient number of strain-stress elastic slopes (data) as shown in Figure 1.e, an equation system can be solved that yields the five elastic constants. However, it is necessary to fix the gauges in the correct *x, y* and *z* positions and directions for each specimen (Figure 1).

This least square method (Amadei, 1996; Cho et al., 2012) can solve a system of equations in which, knowing the values of the strains, the five elastic constants are calculated by solving the equation system. It is understood this approach looks for a mathematical best-fit approach, not accounting for the physical nature of the processes under study.

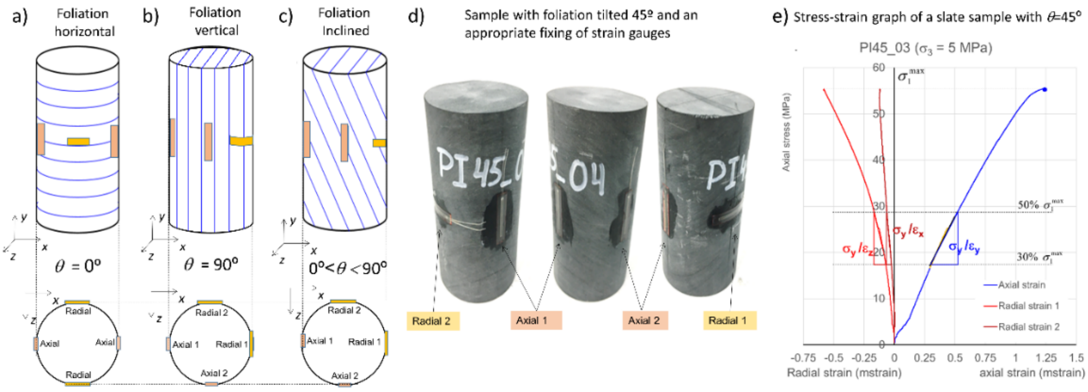

Figure 1. a) Representation and strain gauge configuration for samples with θ=0º, b) same for θ=90º and c) same for samples with 0º<θ<90º, d) actual sample with θ=45º with glued strain gauges (different views) and e) Stress-strain curves of a tests with θ=45º, confined 5 MPa. Adapted from Alejano et al. (2021).

4 Samples

3 Samples

$$\begin{vmatrix} \varepsilon_{\theta=0(x)}/\partial_y \\ \varepsilon_{\theta=0\,(y)}/\partial_y \\ \varepsilon_{\theta 1(x)}/\partial_y \\ \varepsilon_{\theta 1(y)}/\partial_y \\ \varepsilon_{\theta 1(z)}/\partial_y \\ \varepsilon_{\theta 2(x)}/\partial_y \\ \varepsilon_{\theta 2(y)}/\partial_y \\ \varepsilon_{\theta 2(z)}/\partial_y \\ \hdashline \varepsilon_{\theta_3(x)}/\partial_y \\ \varepsilon_{\theta_3(y)}/\partial_y \\ \varepsilon_{\theta_3(z)}/\partial_y \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ sin^2 2\theta_1/4 & sin^2 2\theta_1/4 & 0 & -cos^4 2\theta_1/4 - sin^4 2\theta_1/4 & sin^2 2\theta_1/4 \\ sin^4\theta_1 & cos^4\theta_1 & 0 & -2sin^2 2\theta_1/4 & sin^2 2\theta_1/4 \\ 0 & 0 & -sin^2\theta_1 & -cos^2\theta_1 & 0 \\ sin^2 2\theta_2/4 & sin^2 2\theta_2/4 & 0 & -cos^4 2\theta_2/4 - sin^4 2\theta_2/4 & sin^2 2\theta_2/4 \\ sin^4\theta_2 & cos^4\theta_2 & 0 & -2sin^2 2\theta_2/4 & sin^2 2\theta_2/4 \\ 0 & 0 & -sin^2\theta_2 & -cos^2\theta_2 & 0 \\ \hdashline sin^2 2\theta_3/4 & sin^2 2\theta_3/4 & 0 & -cos^4 2\theta_3/4 - sin^4 2\theta_3/4 & sin^2 2\theta_3/4 \\ sin^4\theta_3 & cos^4\theta_3 & 0 & -2sin^2 2\theta_3/4 & sin^2 2\theta_3/4 \\ 0 & 0 & -sin^2\theta_3 & -cos^2\theta_3 & 0 \end{vmatrix} \times \begin{vmatrix} \frac{1}{E} \\ \frac{1}{E'} \\ \frac{v}{E} \\ \frac{v'}{E'} \\ \frac{1}{G'} \end{vmatrix}$$

Figure 2. Matrix equation analytical systems to obtain 5 elastic parameters based on 3 or 4 sample data.

Another possible approach to compute the five TI elastic constants is based on the Generalized Reduced Gradient (GRG) nonlinear algorithm, as proposed by Alejano et al (2021). This algorithm, as implemented in Excel, can be used to minimize the error on the first strain invariant J_1 (eq. 5):

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III} \tag{5}$$

where ε_x ,ε_y ε_z are the strains measured by the strain gauges on test, and ε_I ,ε_{II} ε_{III} are the analytical solutions in each direction based on computation for every trial set of five elastic constants. The generalized reduced gradient (GRG) method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints. In its basic form, as developed for Excel and used in this study, this solver method controls the slope or gradient of the objective function ─error in strain invariant = $(\varepsilon_x + \varepsilon_y + \varepsilon_z) - (\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})$─ as the input trial values of elastic constants are varied. The method determines that it has reached an optimum solution when the partial derivatives equal zero. So, the approach basically looks for minimizing the sum of errors of the strain invariant (objective function) for all the samples involved. In a certain way, it tends to homogenize the more or less heterogeneous strain data, which makes physical sense. For this application, the GRG approach requires initial parameter values and results can be constrained within specified ranges. The initial values provided are estimated obtaining E' and ν' from average strain results in samples with $\theta = 0$º, E and ν from average strain results in samples with $\theta = 90$º, and the previously computed E' and ν' values, and averaging G from some inclined samples and the previous values. It is relevant to mention that these initial values can be used as first estimates of the TI elastic constants. In our case results were constrained within imposed ranges, namely 1 GPa < E, E' < 100 GPa, 0.01 < ν ,ν' < 0.5 and 1 GPa < G' < 50 GPa. Alejano et al (2021) applied this approach to 54 data sets, though some of these data were incomplete as they corresponded to sets of data with only one or two strain measurements.

3 DATA AND TEST RESULTS

3.1 Original data

The data used in this work were collected from tests performed on a slate from the NW of Spain; further information concerning the rock lithology is available in Alejano et al (2021). To perform the analysis, stress-strain plots of 24 uniaxial and triaxial (with constant confinement) compression tests were selected (as that of Fig. 1.e), arbitrarily among those having a complete set of strain data. It was decided to use eight samples with $\theta = 0°$ and groups of four samples corresponding to θ equal to 30°, 45°, 60° and 75° each. No samples corresponding to $\theta = 90°$ were included due to lack of a sufficient number. The original data are presented in Figure 3 - Table A. Remark for instance that the values of ε_x/σ_y and ε_z/σ_y for the first eight samples ($\theta = 0°$) should be equal but they are not, which is a clear indication of strain heterogeneity or variability that could affect parameter estimates.

Table A. Original database of 24 selected tests.

Sample	Name	Type UCS/TRX	θ	ε_x/σ_y	ε_y/σ_y	ε_z/σ_y
1	PIPE_1	UCS_0	0	-0.0062	0.026	-0.0087
2	PIPE_2	UCS_0	0	-0.0104	0.0221	-0.0108
3	PIPE_3	UCS_0	0	-0.0042	0.0203	-0.0035
4	PIPE_7	TRX_5	0	-0.0102	0.0338	-0.0105
5	PIPE_8	TRX_2.5	0	-0.0113	0.0238	-0.0106
6	PIPE_9	TRX_7.5	0	-0.011	0.0307	-0.0126
7	PIPE_10	TRX_10	0	-0.0115	0.0343	-0.0098
8	PIPE_11	TRX_15	0	-0.0136	0.0253	-0.0151
9	PI30_03	TRX_5	30	-0.0045	0.0282	-0.0075
10	PI30_04	TRX_5	30	-0.0092	0.0284	-0.005
11	PI30_07	TRX_10	30	-0.0133	0.0117	-0.0042
12	PI30_09	TRX_15	30	-0.0085	0.0181	-0.0052
13	PI45_03	TRX_5	45	-0.0081	0.0186	-0.0033
14	PI45_05	TRX_10	45	-0.0076	0.0391	-0.0072
15	PI45_09	TRX_10	45	-0.0068	0.016	-0.0014
16	PI45_10	TRX_15	45	-0.0101	0.0162	-0.0018
17	PI60_02	TRX_5	60	-0.0157	0.0231	-0.0048
18	PI60_06	TRX_10	60	-0.0035	0.0131	-0.0004
19	PI60_07	TRX_15	60	-0.0034	0.0114	-0.0021
20	PI60_08	TRX_15	60	-0.0093	0.0215	-0.001
21	PI75_02	TRX_5	75	-0.0034	0.0147	-0.0014
22	PI75_06	TRX_10	75	-0.0039	0.013	-0.003
23	PI75_08	TRX_15	75	-0.0059	0.0156	-0.0018
24	PI75_09	TRX_15	75	-0.0056	0.0081	-0.004

Table B. Least square estimates for 24 groups of 3 samples.

Comb.	Samples	E	E'	ν	ν'	G'
1	1 9 17	57.00	37.96	0.27	0.25	13.11
2	2 10 18	103.76	38.91	-0.17	0.27	17.85
3	3 11 19	152.22	56.43	0.05	0.36	24.5
4	4 12 20	44.3	33.1	-0.17	0.39	19.65
5	5 13 21	57.66	42.42	0.02	0.34	19.46
6	6 14 22	64.31	29.54	0.28	0.15	11.1
7	7 15 23	54.14	30.43	-0.04	0.31	22.21
8	8 16 24	94.16	40.06	0.14	0.38	19.64
9	1 9 13	122.37	35.77	0.44	0.20	18.43
10	2 10 14	17.76	46.16	0.04	0.44	10.97
11	3 11 15	337.62	55.17	-1.32	0.38	22.57
12	4 12 16	-552.63	31.34	4.46	0.34	23.16
13	5 17 21	71.97	41.92	0.16	0.31	11.49
14	6 18 22	61.59	32.67	0.04	0.29	33.18
15	7 19 23	52.36	29.48	0.02	0.32	41.84
16	8 20 24	85.68	38.4	0.14	0.33	14.89
17	1 9 21	64.86	35.94	0.13	0.18	17.01
18	2 10 22	77.52	41.72	0.22	0.24	13.67
19	3 11 23	64.42	61.14	0.07	0.42	19.56
20	4 12 24	102.73	31.81	0.29	0.30	28.44
21	5 13 17	39.49	43.83	-0.02	0.56	15.74
22	6 14 18	72.38	28.98	0.12	0.18	13.47
23	7 15 19	92.84	29.35	-0.26	0.30	23.65
24	8 16 20	38.78	41.26	-0.21	0.58	18.15

Figure 3. Compiled data information in two tables. Table A: strain slope database according to sample type. Table B: Least square elastic parameter results for groups of 3 samples.

3.2 Least squares and GRG results

To implement suitable equation systems as depicted in Figure 2, it was decided to use the samples in groups of 3 and 4 to start analyzing results. Some problems associated with strain variability and heterogeneity influence on results were detected for two sample groups, and so these groups were not considered. Analytical results are computed based on adapted versions of the matrix in Fig. 2 for groups of 3 and 4 samples. Some possible combinations of the 24 available data sets (each form one test) are used for each grouping of samples (3 and 4) provided there was in each case one and only one sample with $\theta = 0°$ and at least 2 with $\theta \neq 0°$ but different θ. This produced 24 groups of 3 samples (where every sample is represented three times or appears in 3 selected groups, for statistical representativeness) and 16 groups of 4 samples (where every sample with $\theta = 0°$ appears twice and every sample with $\theta \neq 0°$ appears 3 times). The results of the least square computations based on solving matrix equation in Fig. 2 for each sample group by obtaining the inverse matrix in each case is presented in Fig. 3 - Table B for the case of combinations of 3 samples, showing the computed five elastic constants of every group.

Probably due to local inhomogeneity of the strains and strain variability between the different samples, the analytical solution of particular combinations of results produced inaccurate and sometimes senseless results. This includes negative or very high elastic moduli or negative Poisson's ratios, such as in combinations 2, 4, 7, 11, 12, 20, 23 and 24. Interestingly E', ν' or G' do not generally show such variable results. This indicates that basing interpretations on a very small number of tests, even if it is theoretically sound, can produce non-realistic parameters due to strain variability. This can be solved by removing outliers at this stage or by applying other methods.

Based on the method proposed by Cho et al (2012), equation systems are prepared by adding the equations dependent on the angle θ and their corresponding strains in order to apply the least square approach. This is done for the 24 tests and for 2 groups of 12 tests (tests 1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 21 and 22 for the first groups and all the other tests for the second), to obtain complementary results, aiming to help to understand the differences in the approaches. The GRG approach as described above has also been applied to the set of 24 groups.

Table 1 presents the average values and the standard deviation for the least square approach by groups of samples (3, 4, 12 and 24) and the values obtained from applying the GRG nonlinear method to the 24 selected samples in this study and to all samples recovered in a previous analysis (Alejano et al., 2021). It should be noted that, in the first case, average and standard deviation are computed for the case of groups of 3 and 4 samples. This is done for all the values and removing outliers, in which case Young's Moduli less than 1 and more than 100 GPa are removed and neither negative Poisson's ratios or over 0.5 are used.

Table 1. Summary of the average values and standard deviation obtained using different methods.

Method		E (GPa)	E'(GPa)	ν (-)	ν'(-)	G'(GPa)
Least square 3 samples	Average	57.39	38.91	0.20	0.33	19.74
All data	St. dev.	143.70	8.89	0.97	0.11	7.23
Least square 3 samples	Average	68.58	38.91	0.15	0.30	19.74
Outliers removed	St. dev.	25.17	8.89	0.12	0.08	7.23
Least square 4 samples	Average	84.08	38.78	0.09	0.29	18.29
All data	St. dev.	26.17	8.06	0.15	0.09	4.44
Least square 4 samples	Average	75.89	38.78	0.16	0.29	18.29
Outliers removed	St. dev.	14.56	8.06	0.08	0.09	4.44
Least square 12 samples	Average	72.63	37.14	0.06	0.32	18.27
	St. dev.	10.44	1.13	0.09	0.05	4.88
Least square 24 samples	Average	74.90	37.04	0.05	0.32	17.60
GRG 24 Samples	Average	74.13	38.11	0.23	0.31	19.52
GRG Alejano et al (2021)	Average	68.22	38.04	0.23	0.28	19.45

4 DISCUSSION

Looking at the results, it is relevant to note that for all the computation approaches E' and ν', which are ultimately depending on tests with θ=0º, present very stable results: E' = [37.04-38.91] and ν' = [0.29-0.32]. This suggests that when using good quality homogeneous data the computation approach scarcely affects results for these parameters.

However, the values of E and ν, that can be theoretically well tracked if good quality tests with θ= 90º are available, are much more scattered: E = [57.39-84.08] and ν = [0.05-0.23]. This is attributed to the fact that tests with θ=90º were not available in the database (even if tests with θ= 60º or 75º, also strongly influence these parameters), and so these values are less constrained by test results. Additionally, it is observed that strains normal to foliation plane are more heterogeneous which could make these parameters more variable in practice, at least for the slate rocks under scrutiny. The values of G' according to all methods are also rather regular: G' = [17.60-19.74].

The results of the least square matrix solution approach with different number of groups of tests indicate that when a limited number of tests is used, erroneous results appear. Averaging results of these groups of tests can be done to obtain more realistic results for the five TI elastic parameters (preferably removing outliers before averaging). However, it seems that the larger the groups of tests included in the solution matrix, the more reliable obtained parameters were.

The GRG approach also seems to provide good results. Indeed, it needs an initial set of values for the elastic constants, which can be a disadvantage and slightly affects results. It presents the advantage of minimizing the stress invariant for every test, which makes physical sense and tend to merge heterogeneous strain in an apparently more realistic manner.

So it seems that as both methods provide reasonably good approximations, particularly when the set of strain measurements is homogeneous, the least square approach with a good number of tests and the GRG approach can be recommended for transversely isotropic elastic parameter calculations. Similar and more advanced analyses in groups of tests of different TI rocks will be required to confirm the reliability of this recommendation.

5 CONCLUSIONS

A brief study focusing on determining the five elastic constants for transversely isotropic rocks was performed using the data collected on standard uniaxial and triaxial compression tests performed on slate rock samples from NW of Spain. In this study, two main different approaches were used to obtain the elastic constants, the traditional least square solution of a redundant equation system for different groups of test results, and one based on a GRG nonlinear algorithm. The results were compared with those obtained on a previous study (Alejano et al., 2021).

For the rock in question, the values of E', G' and ν' seem to be more regular and easy to derive, however E and ν seem much more variable. This is something attributed to the intrinsically higher variability of these parameters and to the shortage of data related to these parameters in the analyzed suite of tests. As a consequence, it is recommended to use balanced sets of tests including enough results for samples with evenly scattered orientations of the foliation plane.

For homogeneous strain databases, all approaches seem to be reliable. For more heterogeneous ones, the least square method with a large number of results or the GRG approach are preferable. When using the least square approach with small groups of data, removing outliers can contribute to homogenize results. Further research with more data and different rock types is required to better understand how strain heterogeneity affects a proper estimate of the transversely isotropic elastic parameters of foliated rocks.

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