Estimation of fracture diameter probability distribution based on truncated trace-length data

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ABSTRACT: Fracture size is a foundational parameter in the evaluation of mechanical and hydraulic properties of rock masses for constructing geological disposal repository. Washburton's equation and Abel's integral are widely used in estimating the probability density function of fracture diameter, $g(x)$. However, negative values of $g(x)$ appear in some cases because of the use of Abel's integral. Besides, the complete curve of trace-length probability density, $h(y)$, is always needed for estimating $g(x)$, while it is almost impossible to get the whole curve of $h(y)$ because of the truncation error. In this paper, a new method of estimating $g(x)$ based on truncated trace-length data was proposed. Instead of using Abel's integral, $g(x)$ was fitting with a piecewise linear function and thus the problem of negative values of $g(x)$ was also avoided. Case studies with truncated trace-length data indicate that the proposed method is effective in estimating fracture diameter probability distribution.

Keywords: Fracture Diameter, Trace-Length, Probability Density, Washburton's Equation.

1 INTRODUCTION

Fracture size is a foundational parameter in the evaluation of mechanical and hydraulic properties of rock masses (e.g. Dershowitz & Einstein 1988, Chen et al. 2015 and Jing & Stephansson 2007). But identifying the fracture size is still a difficult task, because there exists almost no way of directly measuring the fracture size. In order to estimate the fracture size probability distribution, Washburton (1980) introduced stereological theories, and derived the relationship between the probability density function of diameter, $g(x)$, and the probability density function of trace-length, $h(y)$:

$$h(y) = \frac{y}{m}\int_{y}^{\infty} \frac{g(x)}{\sqrt{x^2 - y^2}} dx \tag{1}$$

where x is fracture diameter; y is fracture trace-length; m is mean value of fracture diameter. In Washburton's equation, fractures are assumed to be flat circular discs of negligible thickness, parallel to each other and non-zero angles to the cutting plane.

With the help of Abel's integral (Santalò 1955), Equation (1) can be transformed as follows:

$$g(x) = -\frac{2mx}{\pi}\int_x^{\infty}\frac{1}{\sqrt{y^2-x^2}}\frac{d\left[\frac{h(y)}{y}\right]}{dy}dx \tag{2}$$

Based on Equation (2), Tonon and Chen (2007) derived the explicit expression of $g(x)$ when $h(y)$ complies with uniform, exponential, gamma or power law. Many researchers got significant achievements in estimating fracture diameter by using Washburton's equation (Zhu et al., 2014; Zhang et al., 2023). However, some difficulties still exist in the use of the above solution. Firstly, when $h(y)$ complies with lognormal law, the explicit expression of $g(x)$ cannot be derived. Secondly, $g(x)$ may give some negative values in some special cases, because the use of Abel's integral releases some constraints in Equation (1).

Meanwhile, $h(y)$ in the above discussion represents the "true" probability density function of trace-length. However, some small fractures are always lost in field investigations due to the resolution limitation. This phenomenon is termed "truncation effect" (Bonnet et al. 2001). Thus measured $h(y)$ from field trace-length data cannot directly used into Equations (1) and (2). At present, no objective method of estimating $g(x)$ has been proposed based on truncated trace-length data from field investigation.

In order to solve the above problems, a new method of estimating fracture diameter probability distribution was proposed by fitting $g(x)$ with a piecewise linear function. The validity of the proposed method was then systematically tested by case studies with truncated trace-length data.

2 EXPLICIT EXPRESSION OF WASHBURTON'S EQUATION

2.1 Derivation

The trace-length probability density function, $h(y)$, can be derived according to Equation (1) when $g(x)$ is known. However, if the expression of $g(x)$ is too complicated, such as the lognormal model, it is difficult to obtain the explicit expression of $h(y)$ (Tonon and Chen 2007). Nevertheless, if $g(x)$ is replaced by a piecewise linear function, the explicit expression of $h(y)$ can be easily obtained. Therefore, we suppose that the $g(x)$ can be fitted by a piecewise linear function as follows:

$$g(x) = \begin{cases} 0, & \text{where } x < x_0 \\ \sum_{j=0}^{1} a_{i,j}\, x^j, & \text{where } x \in [x_i, x_{i+1}), i = 0,1,2,\dots,n-1 \\ 0, & \text{where } x \geq x_n \end{cases} \tag{3}$$

where x_0 and x_n are minimum and maximum fracture diameter respectively; n is piece number of the piecewise function; with the increment of n, Equation (3) can get infinitely close to the true curve of $g(x)$. Based on Equations (1) and (3), the explicit solution of $h(y)$ can be derived as follows:

$$h(y) = \begin{cases} 0, & y \leq 0 \\ \frac{y}{m}\sum_{k=0}^{n-1}\sum_{j=0}^{1} f_j(y, x_{k+1}, x_k, a_{k,j}), & 0 < y \leq x_0 \\ \frac{y}{m}\sum_{j=0}^{1}\left[f_j(y, x_{i+1}, y, a_{i,j}) + \sum_{k=i+1}^{n-1} f_j\big(y, x_{k+1}, x_k, a_{k,j}\big)\right], & y \in (x_i, x_{i+1}] \\ 0, & y > x_n \end{cases} \tag{4}$$

where m is mean diameter and $f_j(\cdot)$ is function as follows:

$$m = \sum_{t=0}^{n-1}\sum_{j=0}^{1}\frac{a_{t,j}}{j+2}\left(x_{t+1}^{j+2} - x_t^{j+2}\right) \tag{5}$$

$$f_0(p_1, p_2, p_3, p_4) = p_4[u(p_1, p_2) - u(p_1, p_3)] \tag{6}$$

$$f_1(p_1, p_2, p_3, p_4) = p_4[v(p_1, p_2) - v(p_1, p_3)] \tag{7}$$

$$u(q, s) = ln\left(\sqrt{s^2 - q^2} + s\right) \tag{8}$$

$$v(q, s) = \sqrt{s^2 - q^2} \tag{9}$$

Based on Eq. (4), the explicit expression of the trace-length cumulative distribution function, $H(y)$, can also be obtained as follows:

$$H(y) = \begin{cases} 0, & y \leq 0 \\ \frac{1}{m}\sum_{k=0}^{n-1}\sum_{j=0}^{1} F_j(y, 0, x_{k+1}, x_k, a_{k,j}) & 0 < y \leq x_0 \\ H(x_0) + \frac{1}{m}\sum_{t=0}^{i-1}\sum_{j=0}^{1} F'_j(x_{t+1}, x_t, x_{t+1}, y, a_{t,j}) & \\ + \frac{1}{m}\sum_{t=0}^{i-1}\sum_{k=t+1}^{n-1}\sum_{j=0}^{1} F_j(x_{t+1}, x_t, x_{k+1}, x_k, a_{k,j}) & \\ + \frac{1}{m}\sum_{j=0}^{1} F'_j(y, x_i, x_{i+1}, y, a_{k,j}) & y \in (x_i, x_{i+1}] \\ + \frac{1}{m}\sum_{k=i+1}^{n-1}\sum_{j=0}^{1} F_j(y, x_i, x_{k+1}, x_k, a_{k,j}) & \\ 1 & y > x_n \end{cases} \tag{10}$$

where $F_j(\cdot)$ and $F'_j(\cdot)$ are functions as follows:

$$F_0(p_1, p_2, p_3, p_4, p_5) = p_5[U(p_1, p_3) - U(p_2, p_3) - U(p_1, p_4) + U(p_2, p_4)] \tag{11}$$

$$F_1(p_1, p_2, p_3, p_4, p_5) = p_5[V(p_1, p_3) - V(p_2, p_3) - V(p_1, p_4) + V(p_2, p_4)] \tag{12}$$

$$F'_0(p_1, p_2, p_3, p_4, p_5) = p_5[U(p_1, p_3) - U(p_2, p_3) - U'(p_1, p_4) + U'(p_2, p_4)] \tag{13}$$

$$F'_1(p_1, p_2, p_3, p_4, p_5) = p_5[V(p_1, p_3) - V(p_2, p_3)] \tag{14}$$

$$U(q, s) = \frac{1}{4}\left[2q^2 ln\left(\sqrt{s^2 - q^2} + s\right) - 2s\sqrt{s^2 - q^2} - q^2 + s^2\right] \tag{15}$$

$$U'(q, s) = \frac{1}{4}[2q^2 ln(q) - q^2] \tag{16}$$

$$V(q, s) = \frac{\sqrt{s^2 - q^2}}{3}(q^2 - s^2) \tag{17}$$

2.2 Verification

In order to verify Eq. (4), some 3D discrete fracture network (DFN) models were generated by using the Monte Carlo method, where $g(x)$ is supposed to comply with the uniform, exponential and lognormal laws respectively. The mean value of fracture diameter for all 3 laws is supposed to be 10m. The standard deviation is supposed to be 5m for the uniform and lognormal laws. The dip, azimuth and volume density of fracture are supposed to be $\pi/2$, 0, and 0.01, and the DFN modeling domain is set as 200m×200m×200m. Then the traces on a horizontal section are investigated and $h(y)$ is then estimated by histogram. The curves of $h(y)$ from DFN models coincide well with that from Eq. (4) (see Figure 1). It indicates that the explicit expression of $h(y)$ obtained by fitting $g(x)$ with a piecewise linear function is effective.

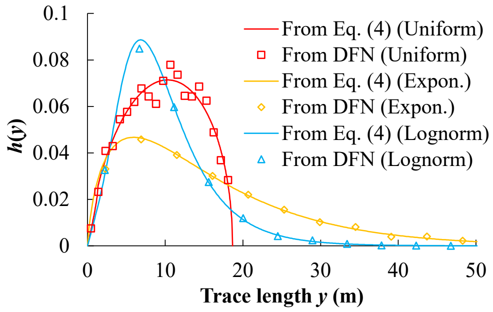

Figure 1. $h(y)$ obtained from Eq. (4) and DFN models under different distribution laws of diameter. (Note: According to Tonon and Chen (2007), different distribution laws were used by different researchers. In order to test the proposed method, 3 typical laws with quite different $g(x)$ curves were used in this paper. Besides, it is impossible to numerically test all the cases found in the real world and several case studies with simple parameters is carried out to just test the feasibility of the proposed method.)

3 NEW METHOD OF ESTIMATING DIAMETER PROBABILITY DISTRIBUTION

3.1 General idea

The cumulative probability of the unmeasured small traces in field investigation can be calculated according to Equation (10):

$$P_{tr} = H(y_{min}) \tag{18}$$

where P_{tr} is actual cumulative probability of the unmeasured small traces; $H(\cdot)$ is cumulative probability function of trace-length; y_{min} is minimum value of trace-length measured in field investigation. Considering the truncation effect, the measured probability density f_t (at $y=y_t$) from field investigation should be corrected as follows:

$$f_{t,real} = (1 - P_{tr})f_t, \qquad t = 1,2,\ldots,N_t \tag{19}$$

where $f_{t,real}$ is true probability density at $y=y_t$ by considering the truncation effect. If we suppose that the fracture diameter complies with one probability distribution law (i.e., uniform, exponential, lognormal, …), $g(x)$ can be reconstructed according to Equation (3) and thus P_{tr} can be calculated according to Equation (10) and (18). Then $f_{t,real}$ can be obtained according to Equation (19). If $\{f_{t,real}, t=1,2,\ldots,N_t\}$ agree well with the calculated $h(y)$ according to Equation (4), the supposed distribution law can be taken as a good estimation of the probability distribution of fracture diameter.

3.2 Detailed steps

Step 1: Calculate average value m_y, standard deviation s_y, minimum value y_{min}, maximum value y_{max}, and probability densities $\{f_t, t=1,2,..,N_t\}$ of measured trace-length data (y_t, $t=1,2,\ldots,N_t$).

Step 2: Define possible $E(x)$ series as $V_e=[E_1, E_2, \ldots, E_{Ne}]$ and $E_i=i\times y_{max}/N_e$, where $i=1,2,\ldots,N_e$.

Step 3: Define possible $S(x)$ series as $V_s=[S_1, S_2, \ldots, S_{Ns}]$ and $S_j=0.5\times j\times y_{max}/N_s$, where $j=1,2,\ldots,N_s$.

Step 4: Select a distribution law of fracture diameter (i.e., uniform, exponential, lognormal, …). Then carry out the following sub-steps.

Step 4.1: Define distribution parameters (i.e., a and b for uniform law; λ for exponential law; μ and σ for lognormal law) by setting $E(x)=E_i$ and $S(x)=S_j$. Thus $g(x)$ is obtained.

Step 4.2: Calculate $h(y)$ and $H(y)$ according to Eq. (4) and (10).

Step 4.3: Calculate P_{tr} by Eq. (18) and $\{f_{t,real}, t=1,2,..,N_t\}$ by Eq.(19).

Step 4.4: Calculate difference between $h(y_i)$ and $\{f_{i,real}, t=1,2,..,N_t\}$ as follows:

$$diff_{i,j} = \sum_{t=1}^{N_t} \left(f_t - f_{t,real}\right)^2 \quad (20)$$

Step 4.5: Repeat Step 4.1~4.4 and calculate $diff_{i,j}$ for each group of (E_i, S_j).

Step 4.6: Search minimum value $diff_{min}$ in $\{diff_{i,j}, i=1,2,\ldots,N_e \text{ and } j=1,2,\ldots,N_s\}$.

Step 5: Repeat Step 4 with different probability distribution laws.

Step 6: Compare $diff_{min}$ from different distribution laws, and the law gives the minimum $diff_{min}$ is taken as the optimal estimation of fracture diameter probability distribution.

3.3 Case studies

Some case studies were carried out with the trace-length data obtained from DFN models generated in Section 2.2, where the diameter was supposed to comply with uniform, exponential and lognormal laws respectively. For each law, 6 trace-length data groups were generated by truncating the original data at 0, 1, 2, 3, 4 and 5m respectively. Then, the diameter distribution law was estimated based on the above truncated data by the proposed method (see Figure 2).

If the true probability distribution law of diameter is uniform law, the estimated $g(x)$ can reflect the form of uniform law (see Figure 2a). The estimated mean value of fracture diameter ranges from 9.37 to 10.00. It means an underestimation of about 0~6.3%. If the true diameter probability distribution law is exponential law or lognormal law, the estimated results coincide well with the true $g(x)$ (see Figure 2b and 2c). In general, the proposed method is effective with truncated trace-length data at 0~5m, when the mean value of fracture diameter is supposed to be 10m.

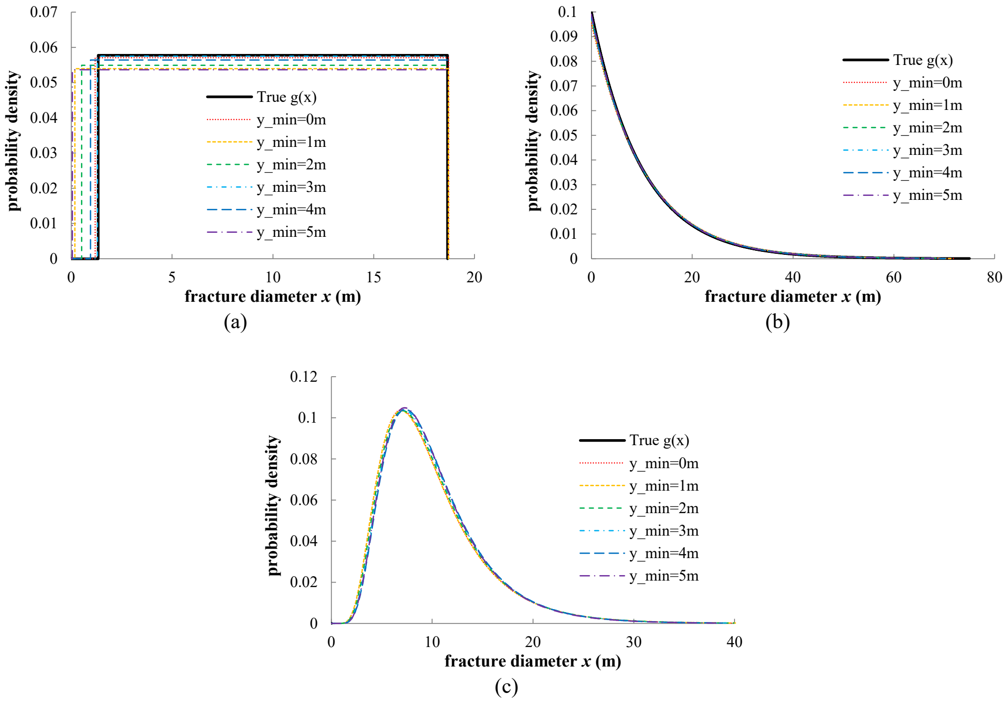

Figure 2. Estimated $g(x)$ of fracture diameter by the new method in comparison with the true $g(x)$, which complies with (a) uniform, (b) exponential and (c) lognormal law respectively.

4 CONCLUSION

The main contribution of this paper is as follows:

1. An explicit expression of Washburton's equation was derived by fitting the probability density function of fracture diameter with a piecewise linear function.
2. A method of estimating fracture diameter distribution from truncated trace-length data was proposed. Case studies proved the validity of the proposed method when the diameter complies with uniform, exponential or lognormal law.
3. 3 different probability distribution laws (uniform, exponential and lognormal) were used in the above cases. Due to the strong adaptability of the piecewise linear function, the proposed method can be easily further developed by adding more distribution laws.

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