An iterative scheme for the determination of the conformal mapping coefficients used in closed-form solutions for tunnels with arbitrary geometry

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ABSTRACT: In case of tunnels with arbitrary geometries, solutions for stresses and displacements in the tunnel exterior might be derived with the aid of the conformal mapping technique of the complex variable method. Thereby, the physical tunnel domain is mapped onto a fictitious unit circle domain on which the elastic potentials, as part of the final solution, are evaluated. The used mapping function involves complex mapping coefficients. In this paper an overview of analytical solutions for stress and displacements fields around tunnels is provided, from the early Kirsch solution to the solutions involving the complex variable theory and conformal mapping. A possible solution procedure for the determination of these mapping coefficients based on an iterative process including the solution of linear systems of equations is presented. The proposed solution procedure can be utilized for the determination of the mapping coefficients of various conformal mapping functions as defined in different closed-form solutions.

Keywords: analytical solution, complex variable theory, conformal mapping, mapping coefficients.

1 INTRODUCTION

The determination of stress and displacement fields arising from the excavation of underground openings has always been a key topic in tunnel engineering. In comparison to the application of numerical methods, analytical solutions to various rock mechanical problems show the general advantage of a rapid implementation with immediate insights into the effects of various influencing parameters.

Nevertheless, analytical methods are often subjected to limitations in their ability to capture some realistic aspects of various boundary value problems. This is mainly due to simplifications made in the derivation of the governing equations also to reduce mathematical complexity to an acceptable level. These simplifications may involve variables such as the problem geometry, the stress boundary conditions and/or the description of the material behavior. With the aim of narrowing down the gap between the predictions and the observable ground behaviors various improvements have been made to closed form solutions in rock mechanics over the last century.

One improvement referring to the possibility for considering arbitrary tunnel geometries in the analytical solutions for stresses and displacements in the tunnel exterior was initiated by Muskhelishvili (1953). He introduced the concept of conformal mapping to the complex variable theory developed by Kolosov (1909). Various available stress-displacement solutions for tunnels with arbitrary geometries in isotropic and anisotropic grounds rely on the complex variable theory and the method of conformal mapping. However, the implementation of these solutions is accompanied by practical difficulties, e.g. when mathematical optimization problems have to be solved (see Winkler et al. 2023).

In this paper a methodology for the determination of the conformal mapping coefficients is provided by setting up a corresponding equation system that is iteratively solved by a least square approximation and updated by a proposed algorithm of He et al. (2022). Section 2 provides a short review of analytical stress-displacement solutions in rock mechanics covering solutions using the conformal mapping technique. Section 3 discusses the basic principles of conformal mapping. In section 4 the proposed solution procedure is presented and section 5 draws the conclusion.

2 REVIEW ON ANALYTICAL SOLUTIONS

In the context of the plane theory of elasticity, a tunnel excavation may be regarded as the problem of an infinite elastic plate weakened by a circular or arbitrarily shaped hole under different loading conditions. The perhaps most famous solution in rock mechanics for the stress field surrounding a circular cavity considering an isotropic material behavior under uniaxial stress was derived by Kirsch (1898). In 1909, Russian mathematician Kolosov (1909) developed the concept of the complex variable theory initiating the possibility for the study of stress concentrations around arbitrary shaped holes in elastic media. He was the first one to solve the special case of the Kirsch problem with elliptical cutouts expressing stresses and displacements in terms of two analytical functions of a complex variable.

In 1913, Inglis (1913) proposed a more rigorous generalization of the Kirsch solution to holes with elliptical shapes in order to model the destructive influence of cracks in brittle materials. Inglis' solution was later picked up by Griffith (1920) marking the starting point of modern fracture mechanics.

In a published book by Muskhelishvili (1953) he extensively investigated the concepts initialized by Kolosov for two-dimensional elasticity problems putting the complex variable theory into a solid framework. He further introduced the concept of conformal mapping to plane elasticity studies to derive solutions for the "hole in plate" problem with more complex shapes of perforations.

Research on plane problems in anisotropic elastic bodies was largely carried out by various authors, e.g. Green & Taylor (1939), Green & Zerna (1954) and Lekhnitskii (1963) who applied complex variable techniques in their studies.

Gerçek (1997) deployed the conformal mapping technique of the complex variable method to investigate stress concentrations around non-circular tunnels with more conventional shapes as used in mining and civil engineering. The study considers a linear-elastic isotropic medium with an arbitrarily oriented biaxial in situ stress field.

Based on the work of Green & Zerna (1954), Hefny & Lo (1999) studied the influence of elastic parameters and non-hydrostatic far-field stress states on the results for stresses and displacements at the circumference of unlined circular tunnels driven in a cross-anisotropic medium.

Exadaktylos & Stavropoulou (2002) derived a stress-displacement elastic closed-form solution based on the complex variable theory and conformal mapping for non-circular tunnels in an isotropic medium with a non-hydrostatic stress field. As an extension to Gerçek (1997), they included the consideration of an incremental stress release due to tunnel excavation and presented a methodology for the determination of the constant conformal mapping coefficients.

The most comprehensible elastic solution for unlined tunnels with arbitrary cross-sections excavated in transversely isotropic ground and based on the complex variable theory and conformal mapping has been proposed by Tran Manh et al. (2015). Besides taking into account an incremental pressure release factor their solution also allows for the consideration of an arbitrary orientation of the biaxial in situ stress-field and the material's planes of isotropy.

3 CONFORMAL MAPPING

3.1 Complex variable z

Within the complex variable theory, the solutions for stress and displacement fields are derived from elastic potential functions, respectively their derivatives (Green & Zerna 1954). Due to difficulties in evaluating these functions at points in the geometrically complex physical tunnel domain, the system is conformally mapped into a fictitious unit circle domain. The values of the potential functions are then evaluated at associated points in the unit circle domain. A point p in the physical tunnel domain, designated as the z-plane, is represented by the complex variable z_p. Variable z_p dependents on Cartesian coordinates x_p and y_p by

$$z_p = x_p + iy_p \tag{1}$$

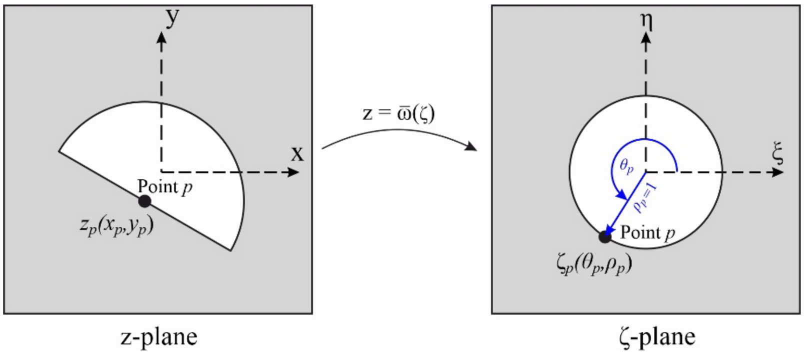


Figure 1. Conformal mapping of points from the physical z-plane onto the unit circle exterior in the ζ –plane.

3.2 Mapping function

A point ζ_p on the unit circle domain (ζ-plane) is associated with a point z_p on the z-plane by the conformal mapping function ϖ (see Figure 1).

$$z_p = \varpi(\zeta_p) \tag{2}$$

While in literature various definitions of the conformal mapping function are defined, exemplarily in this paper the following definition is used based on the Laurent series (Tran Manh et al. 2015), describing a conformal map of the tunnel exterior to the unit circle exterior.

$$z = \varpi(\zeta) = R\left[\zeta + \sum_{n=1}^{N} M_n \zeta^{-n}\right] \tag{3}$$

$$with\ \zeta = \rho e^{i\theta} and\ e^{\pm i\theta n} = \cos(n\theta) \pm i \sin(n\theta)$$

Polar angle θ and polar distance ρ describe the polar coordinates of points on the ζ-plane corresponding to associated points on the z-plane. Parameter N in Eqn. (3) represents the number of terms used in the series expansion. N is typically chosen as three, however, a higher number of terms can be used to increase the mapping accuracy (Exadaktylos & Stavropoulou 2002, Tran Manh et al. 2015, Xiong et al. 2022). R denotes a constant factor relating to the overall size of the original cross section and $M_n = a_n + ib_n$ are complex constant coefficients (Gerçek 1997). R and M_n are the unknowns to be determined, for instance in an iterative manner, focusing on the tunnel boundary C with $\rho = 1$ (unit circle) only.

4 SOLUTION PROCEDURE

In order to solve for the mapping coefficients, the tunnel contour C in the z-plane needs to be discretized into a finite number of M points z_m (m = 1, 2,…, M). For an exemplarily chosen semi-circular cross-section, an increased point density in corners of boundary C on the z-plane is used as suggested by Exadaktylos et al. (2003) for regions with large variations of the radius of curvature (Figure 2a). Each of the discrete points z_m from the z-plane is associated with a corresponding point ζ_m (m = 1, 2,…, M) on the tunnel contour in the ζ-plane for which the position is not known in advance. In case of an infinitely accurate mapping relationship, any point ζ_m from the tunnel contour in the ζ-plane mapped onto the z-plane (image of ζ_m designated as $z_{c,m}$) must be equal to the associated discrete point z_m on the real tunnel contour. Following this ideal consideration, an equation system needs to be set up as described in section 4.1 which can be solved for the sought mapping coefficients.

4.1 Equation system and initial solution

For setting up the system of equations as part of an optimization problem it is helpful to express Eqn. (3) in terms of the two parametric functions as stated in Eqn. (4). The coordinates $x_{c,m}$ and $y_{c,m}$ (m = 1, 2,…, M) represent the Cartesian coordinates of the z-plane image $z_{c,m}$ of a point ζ_m from the tunnel boundary on the ζ-plane (ρ = 1).

$$\begin{aligned} x_{c,m}(\zeta) &= R\left[\rho\, cos(\theta_m) + \sum_{n=1}^{N} \rho^{-n}\,(a_n\, cos(n\theta_m) + b_n\, sin(n\theta_m))\right] \\ y_{c,m}(\zeta) &= R\left[\rho\, sin(\theta_m) - \sum_{n=1}^{N} \rho^{-n}\,(a_n\, sin(n\theta_m) - b_n\, cos(n\theta_m))\right] \end{aligned} \tag{4}$$

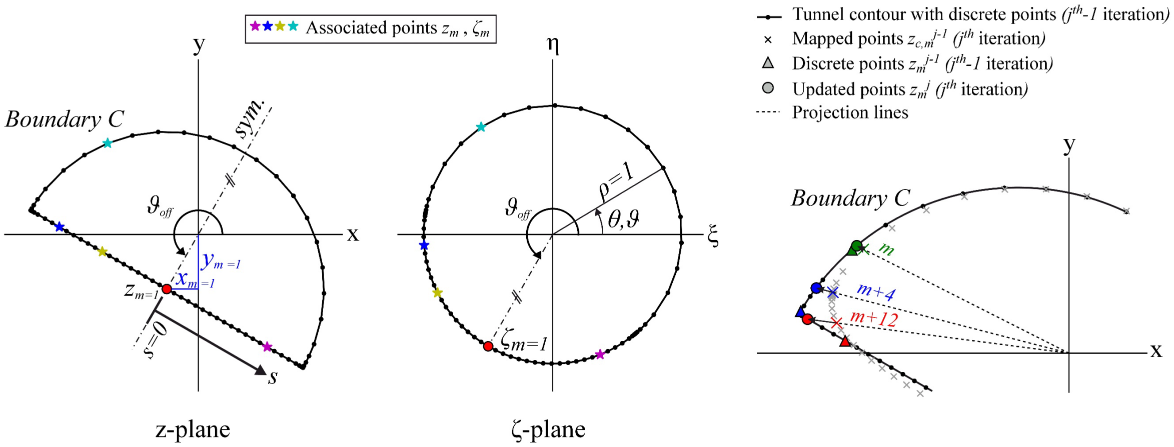
By equalizing the known x- and y-coordinates of each of the M points from the discretized tunnel boundary in the z-plane with the images of the points from the ζ-plane, represented by the functions in Eqn. (4), the following overdetermined non-linear system of equations is received

$$\begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_M \\ y_M \end{bmatrix} = \begin{bmatrix} cos\,\theta_1 & cos\,1\theta_1 & sin\,1\theta_1 & \cdots & cos\,N\theta_1 & sin\,N\theta_1 \\ sin\,\theta_1 & -sin\,1\theta_1 & cos\,1\theta_1 & \cdots & -sin\,N\theta_1 & cos\,N\theta_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ cos\,\theta_M & cos\,1\theta_M & sin\,1\theta_M & \cdots & cos\,N\theta_M & sin\,N\theta_M \\ sin\,\theta_M & -sin\,1\theta_M & cos\,1\theta_M & \cdots & -sin\,N\theta_M & cos\,N\theta_M \end{bmatrix} \cdot \begin{bmatrix} R \\ R\cdot a_1 \\ R\cdot b_1 \\ \vdots \\ R\cdot a_N \\ R\cdot b_N \end{bmatrix} \tag{5}$$

The problem in the form $\boldsymbol{b} = \boldsymbol{A}\mathbf{x}$ as stated in Eqn. (5) consists of finding approximate solutions for θ_m (m = 1, 2,…, M), R, a_n and b_n (n = 1, 2,…,N) to minimize an objective function by defining suitable constraints on these design variables. In literature, the mixed penalty function method is sometimes used to solve this non-linear constrained optimization problem (Zeng et al. 2019). However, it is possible to simplify and transform the non-linear problem into a linear problem by making assumptions on the values for θ_m, presuming that the points ζ_m (ρ=1,θ_m) (m=1, 2,…,M) with $\zeta_m \neq \zeta_{m+1}$ must be distributed over an interval [0,2π] on the unit circle.

The polar angles θ_m are equated with $\vartheta_m = s_m/2\pi+\vartheta_{off}$, describing the normalized path coordinate s_m of a corresponding point z_m along the initially discretized tunnel boundary C including a considered offset angle ϑ_{off} equal to the polar angle of $z_{m=1}$. As can be seen in Figure 2a), the path coordinate s is chosen to start from z_1 at the symmetry axis of the considered semi-circular cross section.

After setting θ_m in Eqn. (5) to constant values ϑ_m, an initial solution for the mapping coefficients can be found by applying a least squares approximation and minimizing the objective function in Eqn. (6) down to a prescribed error tolerance ε. This equation describes the sum of the distance residuals between mapped points $z_{c,m}$ from the unit circle onto the z-plane and discrete points z_m on the tunnel boundary C.



a) Polar angle correspondence between z- and ζ-plane b) Iterative point update on tunnel boundary

Figure 2. a) Polar angle correspondence for points ζ_m in the ζ-plane based on the normalized path coordinate s of points z_m in the the z-plane and b) Update of tunnel boundary points z_m based on the principle of corresponding point polar angle Equality (PCPPAE) after He et al. (2022) by polar projection of points $z_{c,m}$ onto boundary C within a single iteration step j (Exemplarily shown for three points with varying index m).

$$f(R, M_n) = \sum_{m=1}^{M} \sqrt{\left(x_m - x_{c,m}(R, M_n)\right)^2 + \left(y_m - y_{c,m}(R, M_n)\right)^2} \leq \varepsilon \tag{6}$$

Following the assumption on fixed polar angles θ_m for points on the unit circle, the initial solution of the above stated problem does not result in an optimum solution for the mapping coefficients. To improve the mapping accuracy, an iterative procedure acc. to He et al. (2022) is utilized in this paper.

4.2 Iterative procedure

The iterative procedure for the determination of the mapping coefficients starts after an initial solution to the problem, as described in section 4.1, is found. The procedure is based on the principle of corresponding point polar angle equality (PCPPAE), stating that the z-plane polar angles of mapped points $z_{c,m}$ and discrete points z_m on the tunnel boundary are considered to be quasi-equal provided the mapping relationship is sufficiently accurate.

Consequently, in each iteration step j, the points z_m^j are updated by projecting the image points $z_{c,m}^{j-1}$ onto the physical tunnel boundary C via projection lines connecting points $z_{c,m}^{j-1}$ with the coordinate origin. A graphical representation of this principle is given in Figure 2b). The points obtained are used to replace the left-hand side in Eqn. (5) and the updated mapping coefficients are obtained from the described least squares approximation in section 4.1. Iterations j are to be carried out until the mapping function in Eqn. (3) sufficiently satisfies the tunnel boundary approximation.

5 CONCLUSIONS

An iterative procedure for the determination of the conformal mapping coefficients required in many complex variable solutions has been presented. An overdetermined system of non-linear equations has been defined, relating points on the physical tunnel boundary with boundary points on the unit circle plane. Thereby, an exemplary definition of the conformal mapping function based on the Laurent series has been considered. The non-linear system has been transformed into a system of linear equations by making assumptions on the z-plane and ζ-plane point polar angle correspondence. The application of an iterative algorithm by He et al. (2022) has been suggested to find an approximate solution for the conformal mapping coefficients.

The defined procedure serves as a robust computer implementation technique for the determination of the conformal mapping coefficients as required in various plane elasticity complex variable solutions. The accuracy of the determined mapping coefficients can be increased by increasing the number of terms in the series expansion of the mapping function or the number of iteration steps in the solution process.

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